From quantum affine symmetry to the boundary Askey–Wilson algebra and the reflection equation

2008 J. Phys. A: Math. Theor. 41 135201

(http://iopscience.iop.org/1751-8121/41/13/135201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 135201 (18pp) doi:10.1088/1751-8113/41/13/135201

From quantum affine symmetry to the boundary Askey–Wilson algebra and the reflection equation

B Aneva[1], **M Chaichian**[2] **and P P Kulish**[3]

[1] INRNE, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria
[2] High Energy Physics Division, Department of Physical Sciences, University of Helsinki and Helsinki Institute of Physics, PO Box 64, 00014 Helsinki, Finland
[3] St. Petersburg Department of Steklov Institute of Mathematics, Fontanka, 27, St. Petersburg 191023, Russia

Received 27 November 2007
Published 14 March 2008
Online at stacks.iop.org/JPhysA/41/135201

Abstract
Within the quantum affine algebra representation theory, we construct linear covariant operators that generate the Askey–Wilson algebra. It has the property of a coideal subalgebra, which can be interpreted as the boundary symmetry algebra of a model with quantum affine symmetry in the bulk. The generators of the Askey–Wilson algebra are implemented to construct an operator-valued K-matrix, a solution of a spectral-dependent reflection equation. We consider the open driven diffusive system where the Askey–Wilson algebra arises as a boundary symmetry and can be used for an exact solution of the model in the stationary state. We discuss the possibility of a solution beyond the stationary state on the basis of the proposed relation of the Askey–Wilson algebra to the reflection equation.

PACS numbers: 02.30.Ik, 11.30.Na, 05.50.+k, 05.70.Ln

1. Introduction

Quantum affine symmetries [1–5] are intensively developed as rich mathematical structures linking various branches of mathematical physics, such as topological quantum field theories, integrable lattice models of statistical physics, rational conformal theories. They are implemented with the ultimate goal to understand and explore the consequences of the symmetries for a description of physical systems.

Our work concerns the application of quantum symmetries for the exact solvability of many particle lattice systems interacting with stochastic dynamics.

The main idea of integrability of lattice systems (within the inverse scattering method [6]) is the existence of a family of commuting transfer matrices, depending on a spectral parameter. For quantum spin chains, the transfer matrices give rise to infinitely many mutually commuting conservation laws. This is the Abelian symmetry of the system. The infinitely many commuting conserved charges can be diagonalized simultaneously and their common eigenspace is finite dimensional in most cases. Thus, the Abelian symmetry reduces the degeneracies of the spectrum from infinite to finite which is the reason for integrability. In addition, many systems possess non-Abelian symmetries. They determine the R-matrix operator, a solution of the Yang–Baxter equation, up to an overall scalar factor and are identified as the quantum bulk symmetries. In the presence of general boundaries, the quantum symmetry and the integrability of the model as well are broken. However with suitably chosen boundary conditions [7, 8] a remnant of the bulk symmetry may survive and the system possesses hidden boundary symmetries, which determine a K-matrix, a solution of a boundary Yang–Baxter equation and allow for the exact solvability. Such nonlocal boundary symmetry charges were originally obtained for the sine Gordon model [9] and generalized to affine Toda field theories [10], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [11] or analogously as the one boundary Temperley–Lieb algebra centralizer in the 'nondiagonal' spin-$1/2$ representation [12].

In this paper we consider an algebraic prescription to construct two operators, possessing a coideal property with respect to the quantum affine $U_q(\hat{sl}(2))$. We show that these operators generate an Askey–Wilson algebra which thus turns to be a coideal subalgebra of the quantum affine $U_q(\hat{sl}(2))$. We argue that one can construct a K-matrix in terms of the Askey–Wilson algebra generators, which satisfies a boundary Yang–Baxter equation (known as a reflection equation). As an example of an Askey–Wilson boundary symmetry we consider a model of nonequilibrium physics, the open asymmetric exclusion process with most general boundary conditions. This model is exactly solvable in the stationary state within the matrix product ansatz to stochastic dynamics, and it can be shown that the boundary operators generate the Askey–Wilson algebra. The model is equivalent to the integrable spin-$1/2$ XXZ chain with most general boundary terms, whose bulk Hamiltonian (infinite chain) possesses the quantum affine symmetry $U_q(\hat{sl}(2))$.

2. The quantum affine $U_q(\hat{sl}(2))$

In this section, we recall the definition of the affine $U_q(\hat{sl}(2))$ [2, 3, 13]. We fix a real number $0 < q < 1$ (in the general case, q is complex) and we use the q-symbol in the form

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \equiv [x]_{q^{1/2}}. \tag{1}$$

The quantum affine $U_q(\hat{sl}(2))$ is defined as the associative algebra with a unit with generators $E_i^{\pm}$ and q^{H_i}, $i = 0, 1$, in the Chevalley basis and defining relations

$$\begin{aligned} &q^{H_i} q^{-H_i} = q^{-H_i} q^{H_i} = 1 \\ &q^{H_0} q^{H_1} = q^{H_0+H_1} = q^c \end{aligned} \tag{2}$$

$$\begin{aligned} &q^{H_i} E_i^{\pm} q^{-H_i} = q^{\pm 1} E_i^{\pm} \\ &q^{H_i} E_j^{\pm} q^{-H_i} = q^{\mp 1} E_j^{\pm} \\ &\left[E_i^+, E_j^-\right] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q^{1/2} - q^{-1/2}} \end{aligned} \tag{3}$$

together with the q-Serre relations

$$\left(E_i^{\pm}\right)^3 E_j^{\pm} - [3]\left(E_i^{\pm}\right)^2 E_j^{\pm} E_i^{\pm} + [3] E_i^{\pm} E_j^{\pm} \left(E_i^{\pm}\right)^2 - E_j^{\pm} \left(E_i^{\pm}\right)^3 = 0, \qquad i \neq j, \tag{4}$$

where $[3] = 1 + q + q^{-1}$. The element $c = H_0 + H_1$ is central and its value is the level of the affine $U_q(\hat{sl}_2)$. The algebra is endowed with the structure of a Hopf algebra. Namely, the coproduct Δ, the counit ϵ and the antipod S are defined as

$$\begin{aligned} \Delta\left(E_i^+\right) &= E_i^+ \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^+ \\ \Delta(E_i^-) &= E_i^- \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^- \\ \Delta(H_i) &= H_i \otimes I + I \otimes H_i \end{aligned} \tag{5}$$

$$\epsilon\left(E_i^+\right) = \epsilon(E_i^-) = \epsilon(H_i) = 0, \qquad \epsilon(I) = 1 \tag{6}$$

$$S\left(E_i^\pm\right) = -q^{\mp 1/2} E_i^\pm, \qquad S(H_i) = -H_i, \qquad S(I) = 1. \tag{7}$$

Let α_0 denote the longest root and ρ is $1/2$ the sum of positive roots. We consider the $U_q(\hat{sl}(2))$ algebra with a scaling element d, defined by $(d, \alpha_0) = 1$ ((,) is the nondegenerate bilinear form on the Cartan subalgebra) and denote $h = (\alpha_0, \alpha_0) + 2(\rho, \alpha_0)$. With a finite-dimensional representation π_V of $U_q(\hat{sl}(2))$, one associates the quantum R-matrix $R^{VV}(\lambda)$ which acts in $V \otimes V$ and satisfies the Yang–Baxter equation. One also has $[d \otimes d, R] = 0$. The universal R-matrix $R(\lambda)$ is uniquely defined [14] by the first terms in its expansion in powers of the Chevalley generators of $U_q(\hat{sl}(2))$:

$$R = q^{c\otimes d + d\otimes c + \sum_{i=0}^{1} H_i \otimes H^i} \left(1 \otimes 1 + (q^{1/2} - q^{-1/2}) \sum_{i=0}^{1} q^{H_i/2} E_i^+ \otimes q^{-H_i/2} E_i^- + \cdots \right). \tag{8}$$

One can define an automorphism T_λ:

$$T_\lambda H_i = H_i, \qquad T_\lambda E_i^\pm = \lambda^{\pm 1} E_i^\pm, \qquad i = 0, 1, \tag{9}$$

and put $R(\lambda) = (T(\lambda) \otimes \mathrm{id})R$. For a fixed finite-dimensional representation (π, V) of the quotient algebra $U_q(\hat{sl}(2))$, obtained by setting $c = 0$, one has $R^{VV}(\lambda) = (\pi \otimes \pi)R(\lambda)$. Following [14], one introduces the currents $L^\pm(\lambda) \in EndV \otimes U_q(\hat{sl}(2))$, given by $L^+ = (\mathrm{id} \otimes \pi_V)(R), L^- = (id \otimes \pi_V)(R^t)$, which are explicitly expressed in terms of the Chevalley generators of $U_q(\hat{sl}(2))$. The operators $L^\pm(\lambda)$ generate a Hopf algebra $A(R)$ and their matrix coefficients generate an algebra $A_0(R) \subset A(R)$. Let $L(\lambda)$ denote the quantum current

$$L(\lambda) = L^+(\lambda q)(L^-(\lambda)^{-1} \tag{10}$$

with a finite Laurent series expansion

$$L(\lambda) = \sum_n l_V(n)\lambda^{-n-2}, \qquad n \in Z. \tag{11}$$

A theorem (by Reshetikhin and Semenov–Tian–Shansky [14]) states that the element $t(\lambda) = \mathrm{tr}_q(L(\lambda))$ lies in the centre of the quotient algebra of $A(R)$, obtained by setting $c = -h$. Hence from the explicit expressions of the currents $L^\pm$ in terms of the Chevalley generators follows that $t(\lambda)$ is the generating function of the Casimir elements of the quotient algebra of $U_q(\hat{sl}(2))$, obtained by setting $c = -h$.

For our purposes we will need a slightly different realization of the algebra in terms of the Chevalley generators and following [15] we define a new basis in $U_q(\hat{sl}(2))$ generated by $H_i, Q_i^s, \bar{Q}_i^s$:

$$Q_i^s = (q^{1/2} - q^{-1/2})E_i^+ q^{-H_i/2} + q^{-H_i}, \qquad \bar{Q}_i^s = -(q^{1/2} - q^{-1/2})E_i^- q^{-H_i/2} + q^{-H_i}. \tag{12}$$

Let now u, u^*, v, v^* be some (complex) scalars. We denote

$$U = uQ_0^s, \qquad U^* = u^* Q_1^s, \qquad V = v\bar{Q}_0^s, \qquad V^* = v^* \bar{Q}_1^s. \tag{13}$$

Then we have

$$q^{1/2}VU - q^{-1/2}UV = (q^{1/2} - q^{-1/2})vu \tag{14}$$

$$q^{1/2}UV^* - q^{-1/2}V^*U = (q^{1/2} - q^{-1/2})uv^*q^{-c} \tag{15}$$

$$q^{1/2}V^*U^* - q^{-1/2}U^*V^* = (q^{1/2} - q^{-1/2})v^*u^* \tag{16}$$

$$q^{1/2}U^*V - q^{-1/2}VU^* = (q^{1/2} - q^{-1/2})u^*vq^{-c}. \tag{17}$$

The operators U, U^*, V, V^* satisfy the following q-Serre relations which are the direct consequence of (4):

$$U^3U^* - [3]U^2U^*U + [3]UU^*U^2 + U^*U^3 = 0 \tag{18}$$

$$U^{*3}U - [3]U^{*2}UU^* + [3]U^*UU^{*2} + UU^{*3} = 0 \tag{19}$$

$$V^3V^* - [3]V^2V^*V + [3]VV^*V^2 + V^*V^3 = 0 \tag{20}$$

$$V^{*3}V - [3]V^{*2}VV^* + [3]V^*VV^{*2} + VV^{*3} = 0. \tag{21}$$

We can now consider the linear combinations

$$A = U + V, \qquad A^* = U^* + V^*. \tag{22}$$

It can be verified directly by using the q-commutation relations (14)–(17) between the operators U, V, U^*, V^* and the q-Serre relations (18)–(21) that the operators A, A^* satisfy the following relations which are the defining relations of a tridiagonal Askey–Wilson (AW) algebra:

$$\begin{aligned} &A^3A^* - [3]_qA^2A^*A + [3]_qAA^*A^2 + A^*A^3 = -uv(q - q^{-1})^2[A, A^*] \\ &A^{*3}A - [3]_qA^{*2}AA^* + [3]_qA^*AA^{*2} + AA^{*3} = -u^*v^*(q - q^{-1})^2[A^*, A], \end{aligned} \tag{23}$$

where $[3]_q = q + q^{-1} + 1$ and $[X, Y] = XY - YX$. In the following section, we are going to consider a general realization of the Askey–Wilson algebra as a coideal subalgebra of the quantum affine $U_q(\hat{sl}(2))$.

3. The tridiagonal Askey–Wilson algebra

We begin this section with some definitions [16, 17]. Let V be a vector space with (in)finite positive dimension. A tridiagonal pair on V is an ordered pair A, A^* where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy the following conditions: (1) there exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal, (2) there exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal. (A tridiagonal matrix is irreducible whenever all entries on the superdiagonal and subdiagonal are nonzero.) A tridiagonal pair A, A^* on V is a Leonard pair if for each A, A^* all eigenspaces are of dimension 1.

Definition 1. *For a Leonard pair A, A^* on V there exists a sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*, \omega, \eta, \eta^*$, such that*

$$\begin{aligned} &-\beta AA^*A + A^2A^* + A^*A^2 - \gamma\{A, A^*\} - \rho A^* = \gamma^*A^2 + \omega A + \eta \\ &-\beta A^*AA^* + A^{2*}A + AA^{2*} - \gamma^*\{A, A^*\} - \rho^*A = \gamma A^{*2} + \omega A^* + \eta^*, \end{aligned} \tag{24}$$

which is uniquely determined by the pair. Equations (24) are called the Askey–Wilson relations.

Definition 2. *For a tridiagonal pair A, A^* on V there exists a sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*$, such that*

$$\begin{aligned} &[A, -\beta AA^*A + A^2A^* + A^*A^2 - \gamma\{A, A^*\} - \rho A^*] = 0 \\ &[A^*, -\beta A^*AA^* + A^{2*}A + AA^{2*} - \gamma^*\{A, A^*\} - \rho^* A] = 0, \end{aligned} \tag{25}$$

which is uniquely determined by the pair. It has been proved [18] for a Leonard pair that the two definitions are equivalent.

Definition 3. *A tridiagonal algebra is an associative algebra with a unit generated by the pair A, A^* subject to relations (24), respectively (25).*

Affine transformations

$$A \to tA + c', \qquad A^* \to t^*A^* + c^*, \tag{26}$$

where t, t^*, c', c^* are some scalars, act on the operators A, A^* of the Askey–Wilson relations and can be used to bring a tridiagonal pair in a reduced form with $\gamma = \gamma^* = 0$. (The label c' is used to avoid confusion with the $U_q(\hat{sl}(2))$ central element.) Examples are the q-Serre relations with $\beta = q + q^{-1}$ and $\gamma = \gamma^* = \rho = \rho^* = 0$ and the Dolan–Grady relations [19] with $\beta = 2, \gamma = \gamma^* = 0, \rho = k^2, \rho^* = k^{*2}$:

$$[A, [A, [A, A^*]]] = k^2[A, A^*] \qquad [A^*, [A^*, [A^*, A]]] = k^{*2}[A^*, A]. \tag{27}$$

The algebra (24) was first considered by Zhedanov [20] who showed that the Askey–Wilson (AW) polynomials gave rise to two infinite-dimensional matrices satisfying the AW relations. The tridiagonal relations have recently been discussed in a more general framework [16, 17, 21], where tridiagonal pairs have been classified according to their dependence on the sequence of scalars [16, 17] and a correspondence to the orthogonal polynomials in the Askey–Wilson scheme was given. In [21] the AW algebra (24) with $\gamma = \gamma^* = 0$ has been equivalently described as an algebra with two generators and with structure constants determined in terms of the elementary symmetric polynomials in four parameters $a, b, c, d, abcd \neq q^m, m = 0, 1, 2\ldots; q \neq 0, q^k \neq 1, k = 1, 2, \ldots.$

We can now formulate the following statement which defines a homomorphism of the AW algebra to the quantized affine algebra $U_q(\hat{sl}(2))$.

Proposition 1. *Let u, u^*, v, v^*, k, k^* be some scalars. The operators A, A^* defined by*

$$\begin{aligned} A &= uE_0^+q^{-H_0/2} + vE_0^-q^{-H_0/2} + kq^{-H_0} \\ A^* &= u^*E_1^+q^{-H_1/2} + v^*E_1^-q^{-H_1/2} + k^*q^{-H_1} \end{aligned} \tag{28}$$

(it is assumed that $E_i^{\pm}$ in (28) are rescaled by $\pm(q^{1/2} - q^{-1/2})$ according to (12)) and their q-commutator

$$[A, A^*]_q = q^{1/2}AA^* - q^{-1/2}A^*A \tag{29}$$

form a closed linear algebra, the Askey–Wilson algebra:

$$\begin{aligned} &[[A, A^*]_q, A]_q = -\rho A^* - \omega A - \eta \\ &[A^*, [A, A^*]_q]_q = -\rho^* A - \omega A^* - \eta^*, \end{aligned} \tag{30}$$

where the (representation dependent) structure constants are given by

$$-\rho = uv(q - q^{-1})^2, \qquad -\rho^* = u^*v^*(q - q^{-1})^2 \tag{31}$$

$$-\omega = -(q^{1/2} - q^{-1/2})^2\big(kk^* + l_V^0(uu^*q^{1/2} + v^*vq^{-1/2})\big) \tag{32}$$

$$-\eta = (q - q^{-1})(q^{1/2} - q^{-1/2})\big(-k(uu^*q^{1/2} + v^*vq^{-1/2}) - l_V^0 uvk^*\big) \tag{33}$$

$$-\eta^* = (q - q^{-1})(q^{1/2} - q^{-1/2})\big(-k^*(uu^*q^{1/2} + v^*vq^{-1/2}) - l_V^0 u^*v^*k\big). \tag{34}$$

The element l_V^0 is the coefficient $l_V(n)$ to λ^{-n-2}, for $n = 0$, in the Laurent series (11) of the quantum current for any highest weight module over $U_q(\hat{sl}(2))$ and is central (the quadratic Casimir element) in the quotient algebra of $U_q(\hat{sl}(2))$, obtained by setting $c = -h$ (with $h = 1$). On a highest weight irreducible representation Vl_V^0 is a scalar.

We note that equation (28) defines the homomorphism of the AW algebra to the affine algebra $U_q(\hat{sl}(2))$. For $k = u + v, k^* = u^* + v^*$, one recovers the particular case (22). Special cases of this homomorphism are the representation considered by Terwilliger [16] with $\rho = \rho^* = 0$ and the one by Baseilhac and Koizumi [22] with $u = v^*$ and $v = u^*$ in (28). Making use of the evaluation representation for the $U_q(\hat{sl}(2))$ generators in (28)

$$\pi_v\big(E_1^{\pm}\big) = E^{\pm}, \qquad \pi_v\big(E_0^{\pm}\big) = v^{\pm 1}E^{\mp}, \qquad \pi_v(q^{H_1}) = q^H, \qquad \pi_v(q^{H_0}) = q^{-H}, \tag{35}$$

where $E^{\pm}$, H are the $U_q(sl(2))$ generators, we obtain the Granovskii and Zhedanov realization [23].

The algebraic relations (23) and (30) are the two equivalent defining relations of the AW algebra with two generators. The homomorphism (28) defines the AW algebra with two generators as the linear covariance algebra for $U_q(\hat{sl}(2))$ with operator-valued structure constants.

Definition 4. *The AW algebra with two generators A, A^* defined by the homomorphism (28) is a deformation in two parameters ρ, ρ^* of the q-Serre relations of level zero quantum affine $U_q(\hat{sl}(2))$, such that it results in a shift of the central charge to a non zero value $c = -1$.*

Taking the limit $q \to 1$ in the defining relations (23), one obtains a two-parameter deformation of the Serre relations of level zero affine $sl(2)$, known as the Dolan–Grady relations.

From the explicit realization of the operators A, A^* follows that they generate a linear covariance algebra for $U_q(\hat{sl}(2))$, which has the property of a coideal subalgebra. Let $B_q(\hat{sl}(2))$ denote the algebra generated by A, A^*.

Proposition 2. *The Askey–Wilson algebra defined by the homomorphism (28) is a coideal subalgebra of $U_q(\hat{sl}(2))$. The proof is straightforward by using the comultiplication (5). One has*

$$\Delta(A) = I \otimes A + (A - kI) \otimes q^{-H_0} \qquad \Delta(A^*) = I \otimes A^* + (A^* - k^*I) \otimes q^{-H_1}, \tag{36}$$

where the expressions on the RHS of (36) obviously belong to $B_q(\hat{sl}(2)) \otimes U_q(\hat{sl}(2))$.

4. Representations of the Askey–Wilson algebra

The Askey–Wilson algebra is known to possess very important properties which allow to obtain its ladder representations. We briefly sketch these properties (for details, see [16, 20, 21]). Namely, there is a representation with basis f_r with respect to which the operator A is diagonal:

$$Af_r = \lambda_r f_r, \tag{37}$$

where the eigenvalues satisfy a quadratic equation

$$\lambda_r^2 + \lambda_s^2 - (q + q^{-1})\lambda_r\lambda_s - \rho = 0 \tag{38}$$

(which yields two different eigenvalues λ_{r+1} and λ_{r-1} for a fixed eigenvalue λ_r) and the operator A^* is tridiagonal:

$$A^* f_r = a_{r+1} f_{r+1} + b_r f_r + c_{r-1} f_{r-1}. \tag{39}$$

Depending on the sign of ρ, the spectrum of the diagonal operator is hyperbolic of the form sh or ch and exp if $\rho = 0$. The algebra possesses a duality property. Due to the duality property, the dual basis exists with respect to which the operator A^* is diagonal and the operator A is tridiagonal. We have

$$A^* f_p^* = \lambda_p^* f_p^* \qquad A f_s^* = a_{s+1}^* f_{s+1}^* + b_s^* f_s^* + c_{s-1}^* f_{s-1}^*, \tag{40}$$

where λ_p^* satisfies the quadratic equation (38) with $-\rho$ replaced by $-\rho^*$. The overlap function of the two basis $\langle s|r\rangle = \langle f_s^*|f_r\rangle$ can be expressed in terms of the Askey–Wilson polynomials. Let $p_n = p_n(x; a, b, c, d)$ denote the nth Askey–Wilson polynomial [24] depending on four parameters a, b, c, d:

$$p_n(x; a, b, c, d) = {}_4\Phi_3 \begin{pmatrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} & \\ ab, ac, ad & |q; q \end{pmatrix} \tag{41}$$

with $p_0 = 1, x = y + y^{-1}$ and $0 < q < 1$. Then there is a basic representation of the AW algebra [17, 21]:

$$[A, A^2A^* - (q + q^{-1})AA^*A + A^*A^2 + abcdq^{-1}(q - q^{-1})^2 A^*] = 0$$
$$[A^*, A^{*2}A - (q + q^{-1})A^*AA^* + AA^{*2} + (q - q^{-1})^2 A] = 0$$

in the space of symmetric Laurent polynomials $f[y] = f[y^{-1}]$ with a basis $(p_0, p_1, \ldots)$ as follows:

$$Af[y] = (y + y^{-1})f[y], \qquad A^* f[y] = \mathcal{D}f[y], \tag{42}$$

where $\mathcal{D}$ is the second-order q-difference operator [24] having the Askey–Wilson polynomials p_n as eigenfunctions. It is a linear transformation given by

$$\begin{aligned}\mathcal{D}f[y] = (1 + abcdq^{-1})f[y] &+ \frac{(1 - ay)(1 - by)(1 - cy)(1 - \mathrm{d}y)}{(1 - y^2)(1 - qy^2)}(f[qy] - f[y]) \\ &+ \frac{(a - y)(b - y)(c - y)(d - y)}{(1 - y^2)(q - y^2)}(f[q^{-1}y] - f[y])\end{aligned} \tag{43}$$

with $\mathcal{D}(1) = 1 + abcdq^{-1}$. The eigenvalue equation for the joint eigenfunctions p_n reads

$$\mathcal{D}p_n = \lambda_n^* p_n, \qquad \lambda_n^* = q^{-n} + abcdq^{n-1}, \tag{44}$$

and the operator A^* is represented by an infinite-dimensional matrix diag $(\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots)$. The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix. Let $\mathcal{A}$ denote the matrix whose matrix elements enter the three-term recurrence relation for the Askey–Wilson polynomials:

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0 \tag{45}$$

$$\mathcal{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix}. \tag{46}$$

The explicit form of the matrix elements of A reads

$$a_n = a + a^{-1} - b_n - c_n \tag{47}$$

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \tag{48}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \tag{49}$$

The basis is orthogonal with the orthogonality condition for the Askey–Wilson polynomials [24] $P_n = a^{-n}(ab, ac, ad; q)_n p_n$:

$$\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m(x; a, b, c, d|q) P_n(x; a, b, c, d|q)\, \mathrm{d}x = h_n \delta_{mn}, \tag{50}$$

where

$$w(x) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}, \qquad h(x,\mu) = \prod_{k=0}^{\infty}[1 - 2\mu x q^k + \mu^2 q^{2k}]$$

and

$$h_n = \frac{(abcdq^{n-1}; q)_n (abcdq^{2n}; q)_\infty}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}. \tag{51}$$

As noted in the previous section, a tridiagonal pair of operators A, A^* is determined up to the affine transformation. One can appropriately rescale the operators to obtain the algebraic relations in the form

$$\begin{aligned} &[A, A^2A^* - \beta AA^*A + A^*A^2 - (q-q^{-1})^2 A^*] = 0 \\ &[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - (q-q^{-1})^2 A] = 0, \end{aligned} \tag{52}$$

which results in the corresponding rescaling of the matrix elements. A shift of the operators has no other effect but shifting the diagonal elements of the representing matrices.

5. Askey–Wilson algebra and reflection equation

We consider models of statistical physics in which the spin variable is associated with the site i of a one-dimensional lattice. An example of a model with quantum affine symmetry is the spin-1/2 XXZ model with the Hamiltonian defined on an infinite-dimensional chain:

$$H = -\frac{1}{2}\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z\right), \tag{53}$$

where the Pauli matrices $\sigma_i^x, \sigma_i^y, \sigma_i^z$ act on the ith component of the infinite tensor product $\ldots \otimes V_{i-1} \otimes V_i \otimes V_{i+1} \otimes \ldots$, with $V = \mathbf{C}^2$. This model is known to be integrable [25] within the representation theory of the affine quantized algebra $U_q(\hat{sl}(2))$. Namely, given the $U_q(\hat{sl}(2))R$-matrix operator $R(z_1/z_2) \in End_{\mathbf{C}} V_{z_1} \otimes V_{z_2}$, where V_z is the two-dimensional $U_q(\hat{sl}(2))$ evaluation module, satisfying the Yang–Baxter equation

$$R_{12}(z_1/z_2)R_{13}(z_1)R_{23}(z_2) = R_{23}(z_2)R_{13}(z_1)R_{12}(z_1/z_2), \tag{54}$$

then the Hamiltonian is written as $H = \sum H_{ii+1}$, where the two-site Hamiltonian density is obtained as

$$H_{ii+1} = \frac{\mathrm{d}}{\mathrm{d}u} PR_{ii+1}|_{u=0} \tag{55}$$

with P as the permutation operator and $z_1/z_2 = \mathrm{e}^u$. The generators act on the quantum space by means of the infinite coproduct and the invariance with respect to the affine $U_q(\hat{sl}(2))$ manifests in the property

$$[H, \Delta^{\infty}(G_k)] = 0 \tag{56}$$

for any of the generators G_k of $U_q(\hat{sl}(2))$. If we introduce for finite chain a boundary of a particular form, such as diagonal boundary terms, the symmetry is reduced to $U_q(sl(2))$ and the invariant Hamiltonian has the form [26]

$$H_{XXZ}^{QGr} = -1/2\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta_q\sigma_i^z\sigma_{i+1}^z + h\left(\sigma_{i+1}^z - \sigma_i^z\right) + \Delta_q\right), \tag{57}$$

where

$$\Delta_q = -\tfrac{1}{2}(q+q^{-1}), \qquad h = \tfrac{1}{2}(q-q^{-1}). \tag{58}$$

In the presence of a boundary in addition to the R-matrix, there is one more matrix $K(z)$ which satisfies the boundary Yang–Baxter equation, also known as a reflection equation:

$$R(z_1/z_2)(K(z_1)\otimes I)R(z_1z_2)(I\otimes K(z_2)) - (I\otimes K(z_2))R(z_1z_2)(K(z_1)\otimes I)R(z_1/z_2) = 0. \tag{59}$$

Within the quantum inverse scattering method, the K-matrix is related to the quantum current $L = L^+(L^-)^{-1}$ (10) where $L^{\pm} \in End V \otimes U_q(\hat{sl}(2))$. In section 2, the two generators of the Askey–Wilson algebra were constructed as linear covariant objects with the coproduct properties of two-sided coideals of the quantum affine symmetry $U_q(\hat{sl}(2))$. It is suggestive to construct the K-matrix in terms of the AW algebra generators.

Let $R(z)$ be the symmetric trigonometric R-matrix with the deformation parameter $q^{1/2}$:

$$R(z) = \begin{pmatrix} q^{1/2}z - q^{-1/2}z^{-1} & 0 & 0 & 0 \\ 0 & z - z^{-1} & q^{1/2} - q^{-1/2} & 0 \\ 0 & q^{1/2} - q^{-1/2} & z - z^{-1} & 0 \\ 0 & 0 & 0 & q^{1/2}z - q^{-1/2}z^{-1} \end{pmatrix}, \tag{60}$$

acting on the auxiliary tensor product space $V_{z_1} \otimes V_{z_2}$ which carry the fundamental representations of the covariance algebra. Then one can construct an operator $L(z)$ [1] in terms of the $U_q(sl(2))$ generators:

$$L(z) = \begin{pmatrix} zq^{J_3} - z^{-1}q^{-J_3} & (q^{1/2} - q^{-1/2})J_- \\ (q^{1/2} - q^{-1/2})J_+ & zq^{-J_3} - z^{-1}q^{J_3} \end{pmatrix}, \tag{61}$$

acting on the tensor product $V_0 \otimes V_Q$ of the auxiliary space V_0 and the quantum space V_Q which in the general case carry finite-dimensional inequivalent $U_q(sl(2))$ representations. The L-operator satisfies

$$R(z_1/z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R(z_1/z_2), \tag{62}$$

where $L_1 = L \otimes I$ and $L_2 = I \otimes L$. As is known, this relation together with the reflection equation (59) constitute the basic algebraic relations of the inverse scattering method to integrable models.

We are now going to construct a solution to equation (59) in terms of the operators A, A^*.

Proposition 3. *Let A, A^* generate the AW algebra, the linear covariance algebra for $U_q(sl(2))$. Then there exists a reflection matrix $K(z) = K^{op}(z) + K^c(z)$, constructed in terms of the AW algebra generators, where the part K^{op} has the form*

$$K^{op}(z) = \begin{pmatrix} q^{1/2}zA - q^{-1/2}z^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}A^* & -\frac{\sqrt{\rho}}{\sqrt{\rho^*}}(q^{1/2} - q^{-1/2})[A^*, A]_q \\ -\rho^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}(q^{1/2} - q^{-1/2})[A, A^*]_q & -q^{-1/2}z^{-1}A + q^{1/2}z\frac{\sqrt{\rho}}{\sqrt{\rho^*}}A^* \end{pmatrix} \tag{63}$$

and the part $K^c(z)$ is

$$K^c_{11} = \frac{q^{1/2}z\eta^* - q^{-1/2}z^{-1}\eta}{\rho(q^{1/2}+q^{-1/2})}, \qquad K^c_{22} = \frac{q^{-1/2}z\eta - q^{1/2}z^{-1}\eta^*}{\rho(q^{1/2}+q^{-1/2})}$$
$$K^c_{12} = -\rho\frac{q^{1/2}z^2+q^{-1/2}z^{-2}}{q^{1/2}+q^{-1/2}} - \frac{\sqrt{\rho}}{\sqrt{\rho^*}}\omega, \qquad K^c_{21} = -\frac{q^{1/2}z^2+q^{-1/2}z^{-2}}{q^{1/2}+q^{-1/2}} - \rho^{-1}\frac{\sqrt{\rho}}{\sqrt{\rho^*}}\omega. \tag{64}$$

The matrix $K(z)$ is a solution of the boundary Yang–Baxter equation (59) provided the operators A, A^* obey the tridiagonal algebraic relations of the AW algebra in the reduced general form (30) with all structure constants $\rho, \rho^*, \omega, \eta, \eta^*$ nonzero. We denote this solution $K(z,\rho)$.

The proof of this proposition is rather long but straightforward. It is directly verified using the explicit form of the R-matrix (60) and the AW algebraic relations (30) that the boundary matrix K from (63), (64) solves the reflection equation (59).

We emphasize on the factor $\frac{\sqrt{\rho}}{\sqrt{\rho^*}}$ to A^* and ω in the K-matrix. This factor is due to the fact that the solution of the boundary Yang–Baxter equation (59) in terms of the AW algebra generators requires $\rho = \rho^*$. This is not a problem since given the AW algebra in the general form (30), we can relate it to an algebra with $\rho = \rho^*$ rescaling $A^* \to \frac{\sqrt{\rho}}{\sqrt{\rho^*}}A^*$. Alternatively, we can rescale $A \to \frac{\sqrt{\rho^*}}{\sqrt{\rho}}A$ to obtain an AW algebra with $\rho = \rho^*$. This gives a second solution $K(z,\rho^*)$ of the reflection equation. Its $K^{op}(z,\rho^*)$ part has the form

$$K^{op}(z) = \begin{pmatrix} q^{1/2}z\frac{\sqrt{\rho^*}}{\sqrt{\rho}}A - q^{-1/2}z^{-1}A^* - \frac{\sqrt{\rho^*}}{\sqrt{\rho}}(q^{1/2}-q^{-1/2})[A^*,A]_q \\ -\rho^{*-1}\frac{\sqrt{\rho^*}}{\sqrt{\rho}}(q^{1/2}-q^{-1/2})[A,A^*]_q - q^{-1/2}z^{-1}\frac{\sqrt{\rho^*}}{\sqrt{\rho}}A + q^{1/2}zA^* \end{pmatrix}. \tag{65}$$

The matrix elements of the $K^c(z,\rho^*)$ part are obtained from (64) by the interchange $\rho \leftrightarrow \rho^*$. The solution $K(z,\rho^*)$ can be implemented to construct a solution $K^*(z)$ of the dual reflection equation [7, 27] Namely, the matrix $K^*(z) = K^t(z^{-1},\rho^*)$ solves the dual reflection equation (which is obtained from equation (59) by changing $z_{1,2} \to q^{-1/2}z^{-1}_{1,2}$ and $K \to K^t$).

Setting $\rho = \rho^*$ and $\eta = \eta^* = 0$ in (63) and (64), we obtain the K-matrix considered in [28] for such a very particular case of an AW algebra and for the spin-$1/2$ quantum space representation. We note that an AW algebra in the general form with a sequence of scalars $-(q+q^{-1}), \gamma, \gamma^*, \omega, \eta, \eta^*$ cannot be reduced to such a particular algebra with structure constants $-(q+q^{-1}), \rho, \rho, 0, 0, \omega, 0, 0$. There exists an unique affine transformation [29] to only set $\gamma = \gamma^* = 0$ (and simultaneously either $\rho = \eta^* = 0$ or $\eta = \rho^* = 0$).

6. A model of nonequilibrium physics with the boundary Askey–Wilson algebra

Reaction–diffusion processes provide a playground to increase the utility of quantum groups [30]. As a physical example we consider the asymmetric simple exclusion process (ASEP), a model of nonequilibrium physics with rich behaviour and a wide range of applicability [31–34].

The asymmetric exclusion process is an exactly solvable model of a lattice diffusion system of particles interacting with a hard core exclusion, i.e. the lattice site can be either empty or occupied by a particle. As a stochastic process, it is described in terms of a probability distribution $P(s_i,t)$ of a stochastic variable $s_i = 0, 1$ at a site $i = 1, 2, \ldots, L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s'_i during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s,s')\,\mathrm{d}t$. Due to probability conservation, $\Gamma(s,s) = -\sum_{s'\neq s}\Gamma(s',s)$. The rates $\Gamma \equiv \Gamma_{jl}^{ik}, i, j, k, l = 0, 1$, are assumed to be independent of the position in the bulk. For diffusion processes, the transition rate matrix simply becomes $\Gamma_{ki}^{ik} = g_{ik}$. At the boundaries, i.e. sites 1 and L additional processes can take place with rates L_i^j and R_i^j $(i, j = 0, 1)$. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a particle at a site i. Particles hop to the left with probability $g_{01}\,\mathrm{d}t$ and to the right with probability $g_{10}\,\mathrm{d}t$. The event of exchange happens if out of two adjacent sites, one is a vacancy and the other is occupied by a particle. The symmetric simple exclusion process is known as the lattice gas model of particles hopping between nearest-neighbour sites with a constant rate $g_{01} = g_{10} = g$. The partially asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only and partially asymmetric if there is a different nonzero probability of both left and right hopping. The number of particles in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. Phase transitions inducing boundary processes [35] are the most interesting examples (see [36] for a review) when a particle is added with probability $\alpha\mathrm{d}t$ and/or removed with probability $\gamma\mathrm{d}t$ at the left end of the chain, and it is removed with probability $\beta\mathrm{d}t$ and/or added with probability $\delta\mathrm{d}t$ at the right end of the chain. Without loss of generality, we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$.

The time evolution of the model is governed by the master equation for the probability distribution of the stochastic system

$$\frac{\mathrm{d}P(s,t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s,s')P(s',t). \tag{66}$$

It can be mapped to a Schroedinger equation in imaginary time for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = -HP(t), \tag{67}$$

where $H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}$. The ground state of this, in general non-Hermitian, Hamiltonian corresponds to the stationary probability distribution of the stochastic dynamics. The mapping provides a connection to the integrable $SU_q(2)$-symmetric XXZ quantum spin chain with anisotropy $\Delta = \frac{(q+q^{-1})}{2}$, $q = \frac{g_{01}}{g_{10}} \neq 1$ and most general boundary terms.

We consider the model within the matrix-product-state ansatz of stochastic dynamics [36, 37], which was inspired by the inverse scattering method to integrable models. The idea is that one associates with an occupation number s_i at a position i a matrix $D_{s_i} = D_1$ if a site $i = 1, 2, \ldots, L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and the stationary probability distribution is expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra

$$D_1D_0 - qD_0D_1 = x_0D_1 - D_0x_1, \qquad x_0 + x_1 = 0, \tag{68}$$

where $0 < q < 1$ and x_0, x_1 are representation-dependent parameters. The totally asymmetric process corresponds to $q = 0$ and the symmetric process to $q = 1$. The quadratic algebra with no x-terms on the RHS (i.e. $D_1D_0 - qD_0D_1 = 0$) corresponds to a bulk process with reflecting boundaries.

For an open system with boundary processes, the normalized steady weight of a given configuration is expressed as a matrix element in an auxiliary vector space

$$P(s_1, \ldots .s_L) = \frac{\langle w|D_{s_1}D_{s_2}\ldots D_{s_L}|v\rangle}{Z_L}, \tag{69}$$

with respect to the vectors $|v\rangle$ and $\langle w|$, defined by the boundary conditions

$$(\beta D_1 - \delta D_0)|v\rangle = x_0|v\rangle \qquad \langle w|(\alpha D_0 - \gamma D_1) = \langle w|(-x_1). \tag{70}$$

The normalization factor to the stationary probability distribution is

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle. \tag{71}$$

In the following, we set $x_1 = -x_0$.

The advantage of the matrix-product ansatz is that once the representation of the diffusion algebra and the boundary vectors $|v\rangle$ and $\langle w|$ are known, one can evaluate all the relevant physical quantities such as the mean density at a site i, $\langle s_i\rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{L-i}|v\rangle}{Z_L}$, the two-point correlation function $\langle s_i s_j\rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{j-i-1}D_1(D_0+D_1)^{L-j}|v\rangle}{Z_L}$ and higher correlation functions. The current J through a bond between site i and site $i+1$, $J = \frac{\langle w|(D_0+D_1)^{i-1}(D_1D_0-qD_0D_1)(D_0+D_1)^{L-i-1}|v\rangle}{Z_L}$, has a very simple form $J = x_0\frac{Z_{L-1}}{Z_L}$. The matrix D_0+D_1 enters all the expressions and plays the role of a transfer matrix operator.

The algebraic matrix state approach (MPA) is the equivalent formulation of recursion relations derived for the asymmetric exclusion process (ASEP) in earlier works [38, 39], which could not be readily generalized to other models. In most applications, one uses infinite-dimensional representations of the quadratic algebra. Finite-dimensional representations [40, 41] impose a constraint on the model parameters. They may be useful in relation to Bethe ansatz on a ring [42]. The MPA was generalized to many-species models [36, 43, 44] and to dynamical MPA [45].

The matrices D_0, D_1 of the MPA generate an AW algebra with $\rho = \rho^* = 0$ [46]. We call this algebra the bulk Askey–Wilson algebra. For the particular case of only incoming (outgoing) particle at the left (right) end of the chain, the boundary operators satisfy an isomorphic $\rho = \rho^* = 0$ AW algebra which can be solved by shifted q-deformed oscillators [47–49] as they were applied for the ASEP with such particular boundary conditions [50, 51]. In the general case of incoming and outgoing particles at both boundaries there are four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$, and one needs an additional rule to form two linear independent boundary operators acting on the dual boundary vectors. From the quadratic algebra (68), two independent relations for the boundary operators follow:

$$\beta D_1\alpha D_0 - q\alpha D_0\beta D_1 = x_0(\alpha\beta D_1 + \beta\alpha D_0) \tag{72}$$

and

$$\gamma D_1\delta D_0 - q\delta D_0\gamma D_1 = x_0(\delta\gamma D_1 + \gamma\delta D_0). \tag{73}$$

To form two linearly independent operators $B^R = \beta D_1 - \delta D_0$, $B^L = -\gamma D_1 + \alpha D_0$ for a solution to the boundary problem and in order to emphasize the equivalence of the ASEP to the integrable spin-1/2 XXZ, one can use the $U_q(su(2))$ algebra. It is generated by three elements with the defining commutation relations

$$[N, A_\pm] = \pm A_\pm, \qquad [A_+, A_-] = \frac{q^N - q^{-N}}{q^{1/2} - q^{-1/2}} \tag{74}$$

and a central element

$$Q = A_+A_- - \frac{q^{N-1/2} - q^{-N+1/2}}{(q^{1/2} - q^{-1/2})^2}. \tag{75}$$

Relations (72), (73) can be solved by choosing a representation of the boundary operators in the form

$$\begin{aligned}\beta D_1-\delta D_0 &= \frac{x_0\beta}{\sqrt{1-q}}q^{N/2}A_+ - \frac{x_0\delta}{\sqrt{1-q}}A_-q^{N/2} - x_0\frac{-\beta q^{1/2}+\delta}{1-q}q^N + x_0\frac{\beta-\delta}{1-q},\\ \alpha D_0-\gamma D_1 &= \frac{x_0\alpha}{\sqrt{1-q}}q^{-N/2}A_+ - \frac{x_0\gamma}{\sqrt{1-q}}A_-q^{-N/2} + x_0\frac{\alpha q^{-1/2}-\gamma}{1-q}q^{-N} + x_0\frac{\alpha-\gamma}{1-q}.\end{aligned} \tag{76}$$

Separating the shift parts from the boundary operators and denoting the corresponding rest operator parts by A and A^*, we write the left and right boundary operators in the form

$$\beta D_1-\delta D_0 = A + x_0\frac{\beta-\delta}{1-q} \qquad \alpha D_0-\gamma D_1 = A^* + x_0\frac{\alpha-\gamma}{1-q}. \tag{77}$$

Then the operators A and A^* defined by

$$A = \beta D_1-\delta D_0 - x_0\frac{\beta-\delta}{1-q} \qquad A^* = \alpha D_0-\gamma D_1 - x_0\frac{\alpha-\gamma}{1-q} \tag{78}$$

and their q-commutator

$$[A, A^*]_q = q^{1/2}AA^* - q^{-1/2}A^*A \tag{79}$$

satisfy the boundary Askey–Wilson algebra of the open ASEP:

$$\begin{aligned}&[[A, A^*]_q, A]_q = -\rho A^* - \omega A - \eta\\ &[A^*, [A, A^*]_q]_q = -\rho^* A - \omega A^* - \eta^*\end{aligned} \tag{80}$$

with structure constants given by

$$\rho = x_0^2\beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \qquad \rho^* = x_0^2\alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2 \tag{81}$$

$$-\omega = x_0^2(\beta-\delta)(\gamma-\alpha) - x_0^2(\beta\gamma+\alpha\delta)(q^{1/2}-q^{-1/2})Q \tag{82}$$

$$\begin{aligned}\eta &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\beta\delta(\gamma-\alpha)Q + \frac{(\beta-\delta)(\beta\gamma+\alpha\delta)}{q^{1/2}-q^{-1/2}}\right)\\ \eta^* &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\alpha\gamma(\beta-\delta)Q + \frac{(\alpha-\gamma)(\alpha\delta+\beta\gamma)}{q^{1/2}-q^{-1/2}}\right).\end{aligned} \tag{83}$$

One can further use the affine transformation properties of the AW algebra generators to obtain a representation of the boundary ASEP operators from the basic representation of the AW algebra. We summarize the results for the representation of the ASEP boundary operators (for details, see [46]).

There is a representation π in a space with the AW polynomials $p_n = p_n(x; a, b, c, d)$ (41) as the basis

$$(p_0, p_1, p_2, \ldots)^t \tag{84}$$

with respect to which the right boundary operator $D_1 - \frac{\delta}{\beta}D_0 \equiv D_1 + bdD_0$ is diagonal. The representing matrix is diag $(\lambda_0, \lambda_1, \lambda_2, \ldots)$ with the eigenvalues λ_n given by

$$\lambda_n = \frac{q^{1/2}}{1-q}(bq^{-n} + dq^n + 1 + bd). \tag{85}$$

The left boundary operator $D_0 - \frac{\gamma}{\alpha}D_1 \equiv D_0 + acD_1$ is tridiagonal and its representing matrix has the form

$$\pi(D_0 + acD_1) = \frac{q^{1/2}}{1-q}(b\mathcal{A}^t + 1 + ac), \tag{86}$$

where the matrix $\mathcal{A}$ is given by (46). The dual representation π^* has a basis

$$(p_0, p_1, p_2, \ldots) \tag{87}$$

with respect to which the left boundary operator $\pi^*(D_0 + acD_1)$ is diagonal diag $(\lambda_0^*, \lambda_1^*, \ldots)$ with diagonal elements

$$\lambda_n^* = \frac{q^{1/2}}{1-q}(aq^{-n} + cq^n + 1 + ac). \tag{88}$$

The right boundary operator is represented by a tridiagonal matrix

$$\pi^*(D_1 + bdD_0) = \frac{q^{1/2}}{1-q}(a\mathcal{A} + 1 + bd). \tag{89}$$

The ASEP boundary value problem is satisfied with the left and right boundary vectors chosen in the form

$$\langle w| = h_0^{-1/2}(p_0, 0, 0, \ldots.), \qquad |v\rangle = h_0^{-1/2}(p_0, 0, 0, \ldots)^t, \tag{90}$$

where h_0 is a normalization from the orthogonality condition (50). With this choice, the solutions to the boundary eigenvalue equations uniquely relate (in this representation) the four parameters of the Askey–Wilson polynomials with the boundary probability rates

$$a = \kappa_+^*, \qquad b = \kappa_+, \qquad c = \kappa_-^*, \qquad d = \kappa_-, \tag{91}$$

where

$$\begin{aligned} \kappa_\pm &= \frac{-(\beta - \delta - (1-q)) \pm \sqrt{(\beta - \delta - (1-q))^2 + 4\beta\delta}}{2\beta} \\ \kappa_\pm^* &= \frac{-(\alpha - \gamma - (1-q)) \pm \sqrt{(\alpha - \gamma - (1-q))^2 + 4\alpha\gamma}}{2\alpha}. \end{aligned} \tag{92}$$

The expressions on the RHS of equation (92) are the functions of the parameters which define the phase diagram of the ASEP. They have been used in previously known MPA applications where they have always been taken for granted. It is quite remarkable that here they follow from the properties of the Askey–Wilson algebra representations.

It can be further shown that the transfer matrix D_0+D_1 and each of the boundary operators generate isomorphic AW algebras [46]. In the tridiagonal representation, the transfer matrix D_0+D_1 satisfies the three-term recurrence relation of the AW polynomials which was explored in [52] for the solution of the ASEP in the stationary state. The exact calculation of all the physical quantities, such as the current, correlation functions, etc, in terms of the Askey–Wilson polynomials was achieved without any reference to the AW algebra. The ultimate relation of the exact solution in the stationary state to the AW polynomials was possible due to the AW boundary hidden symmetry of the ASEP with most general boundary conditions. The relation of the AW algebra to the K-matrix, determined by (63) and (64), and satisfying the reflection equation, puts a solution beyond the stationary state into perspective.

7. Interpretation of the ASEP boundary operators

As known the open ASEP is related to the integrable spin-1/2 XXZ quantum spin chain through the similarity transformation $\Gamma = -qU_\mu^{-1}H_{XXZ}U_\mu$ [40]. H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain (57) with anisotropy Δ_q and with added nondiagonal boundary terms B_1 and B_L:

$$H_{XXZ} = H_{XXZ}^{QGr} + B_1 + B_L. \tag{93}$$

The transition rates of the ASEP are related to the boundary terms in the following way (μ is a free parameter, irrelevant for the spectrum):

$$B_1 = \frac{1}{2q}\big(\alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\mu\sigma_1^- - 2\gamma\mu^{-1}\sigma_1^+\big)$$
$$B_L = \frac{\big(\beta + \delta - (\beta - \delta)\sigma_L^z - 2\delta\mu q^{L-1}\sigma_L^- - 2\beta\mu^{-1}q^{-L+1}\sigma_L^+\big)}{2q}. \tag{94}$$

It has been shown by Sandow and Schuetz [53] that the bulk-driven diffusive system with reflecting boundaries can be mapped to the spin-$1/2U_q(su(2))$ invariant quantum spin chain. The $U_q(su(2))$ generators satisfying equations (74) and (75) act on the tensor product representation space $(V^2)^{\otimes L}$ as

$$q^{\pm N} = q^{\pm\frac{\sigma_3}{2}} \otimes q^{\pm\frac{\sigma_3}{2}} \otimes \ldots \otimes q^{\pm\frac{\sigma_3}{2}}$$
$$A_\pm = \sum_i q^{\frac{\sigma_3}{4}} \otimes \ldots \otimes q^{\frac{\sigma_3}{4}} \otimes \sigma_i^\pm \otimes q^{-\frac{\sigma_3}{4}} \otimes \ldots \otimes q^{\frac{-\sigma_3}{4}}, \tag{95}$$

where $\sigma_3, \sigma^\pm$ are the Pauli matrices and the index i means that the matrix is associated with the ith site of the chain (ith position in the tensor product). The representation is completely reducible; the product of L spin-$1/2$ representations decomposes into a direct sum of spin j irreducible representations with the maximal highest weight $j = L/2$ decreasing by 1 to $j = 0$ or $j = 1/2$ for even L or odd L. Within the matrix product approach, the bulk process with reflecting boundary conditions is described by a quadratic algebra

$$D_1D_0 - qD_0D_1 = 0, \tag{96}$$

which defines a two-dimensional noncommutative plane with the $SU_q(2)$ action as its symmetry. The operators associated with the bulk ASEP form the two-dimensional comodule of $SU_q(2)$. As a consequence of equation (96), for generic q, a spin j representation of $U_q(su(2))$ can be realized in the space of the q-symmetrized product of $L = 2j$ two-dimensional representations $D_\mu, \mu = 0, 1,$ with basis $D_0^{L-k}D_1^k, k = 0, 1, \ldots, L$. The stationary probability distribution, i.e. the ground state of the $U_q(su(2))$ invariant Hamiltonian H_{XXZ}^{QGr}, corresponds to the q-symmetrizer of the Young diagram with one row and L boxes [54]. The presence of the boundary processes (i.e. the nondiagonal boundary terms in the Hamiltonian) reduces the $U_q(su(2))$ bulk invariance and amounts to the appearance of linear terms in the quadratic algebra. The boundary conditions define the boundary operators which carry a residual symmetry of the process. It is expressed in the fact that the boundary operators are constructed in terms of the $U_q(su(2))$ generators, as seen from the explicit formulae (76). With $A_\pm, N$ being the generators of a finite-dimensional $U_q(su(2))$ representation, it can be verified from equation (76) that $\alpha D_0 - \gamma D_1$ commutes with $H(q)^{QGr}$ and $\beta D_1 - \delta D_0$ commutes with $H(-q^{-1})^{QGr}$, where according to [26]

$$H^{QGr}(-q^{-1}) = -UH^{QGr}(q)U^{-1} \tag{97}$$

and

$$U = \exp\left(\mathrm{i}\frac{\pi}{2}\sum_{m=1}^{L} m\sigma_m^3\right). \tag{98}$$

Thus, the boundary operators constructed as the linear covariant objects of the bulk $U_q(su(2))$ symmetry acquire a very important physical meaning—they can be interpreted as the two nonlocal conserved charges of the open ASEP. Such nonlocal boundary symmetry charges were originally obtained for the sine Gordon model [9] and generalized to affine Toda field theories [10], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [11]. In particular, the left boundary operator $\alpha D_0 - \gamma D_1$ in the finite-dimensional representation (76) is analogous to the one boundary Temperley–Lieb algebra centralizer in the 'nondiagonal' spin-$1/2$ representation [12].

8. Discussion and conclusion

In this paper, we have considered the homomorphism of the Askey–Wilson algebra with two generators into the quantum affine $U_q(\hat{sl}(2))$ algebra. This homomorphism defines the Askey–Wilson algebra as a coideal subalgebra of $U_q(\hat{sl}(2))$. We have constructed an AW operator-valued *K*-matrix which is a solution of the boundary Yang–Baxter equation (reflection equation). We consider the relation of an AW algebra to a solution of the reflection equation to be important for the exact solvability of a physical system with the quantum $U_q(\hat{sl}(2))$ invariance in the bulk and hidden boundary Askey–Wilson algebra symmetry.

As an example of a physical system with boundary AW algebra we consider a model of nonequilibrium physics, the open asymmetric exclusion process with general boundary conditions. This model is equivalent to the integrable spin-$1/2$ *XXZ* chain with nondiagonal boundary terms whose bulk invariance (infinite spin chain) is $U_q(\hat{su}(2))$. The presence of boundaries breaks the bulk quantum affine symmetry of the equivalent quantum spin chain; however, a remnant of the bulk symmetry survives and it is expressed in the possibility of constructing the ASEP boundary operators in terms of the $U_q(\hat{su}(2))$ generators. Thus the boundary operators of the open asymmetric exclusion process generate an Askey–Wilson algebra, which is the hidden boundary symmetry of the process. The exact solution of the ASEP with most general boundary conditions (four boundary probability rates) in the stationary state was obtained [52] in terms of the AW polynomials without reference to the AW algebra. It was emphasized that the solution was ultimately related to the AW polynomials. Such an ultimate relationship is natural from the point of view of the boundary AW algebra. The existence of the reflection matrix $K(z,\rho)$ (and its dual $K^t(z^{-1},\rho^*)$) constructed in terms of the AW algebra generators and satisfying the boundary Yang–Baxter equation is, in our opinion, the deep algebraic property of the open asymmetric exclusion process that may allow for extending its exact solvability beyond the stationary state.

It is important to emphasize the representation dependence of the Askey–Wilson algebra (as well as of the MPA bulk quadratic algebra (68)). Constructed as a coideal subalgebra, it has the property that the structure constants $\rho, \rho^*, \omega, \eta, \eta^*$ carry the information of the corresponding quantum algebra $U_q(\hat{s}u(2))$. The boundary Askey–Wilson algebra whose structure constants depend on the finite-dimensional $U_q(\hat{s}u(2))$ representations is the ASEP hidden symmetry, and this may have an important consequence in relation to Bethe ansatz integrability. The Bethe solution of the open ASEP [55] was achieved through the mapping to the $U_q(su(2))$ integrable *XXZ* quantum spin chain with most general nondiagonal boundary terms, provided a particular constraint on the model parameters was satisfied. Quite surprisingly, the constraint coincides with the condition for a finite-dimensional representation of the Askey–Wilson boundary algebra. The suitably chosen representation-dependent boundary algebra may turn to be the key in relation to Bethe ansatz integrability. For the ASEP the reduction of the bulk invariance gives rise to the boundary symmetry which remains as the linear covariance algebra of the bulk $U_q(\hat{s}u(2))$ symmetry, and one can further employ Bethe ansatz to obtain exact results for the approach to stationarity at large times and to completely determine the spectrum of the transfer matrix. As commented in [56], the way one can satisfy the condition for the Bethe ansatz solution of the ASEP implies additional symmetries. In our opinion, the linear covariance Askey–Wilson algebra of the bulk $U_q(\hat{s}u(2))$, whose generators are interpreted as the two nonlocal conserved charges of the ASEP, is the hidden symmetry behind Bethe ansatz solvability.

It is worth mentioning that the relation of the ASEP (or the equivalent quantum spin chain) boundary algebra to Bethe ansatz integrability is promising from the point of view of Bethe ansatz perspective in string theory. One is interested in closed strings with periodic boundary conditions. However, it is simpler to find the scattering matrix on the infinite line using asymptotic states and bootstrap. Then the spectrum is determined by asymptotic Bethe equations [57, 58] and they are approximate for a system of a finite size. The study of the Askey–Wilson algebra of a system on a ring with periodic boundary conditions, which is interesting on its own, might also be useful for application to strings of finite length.

We have obtained an Askey–Wilson algebra as a coideal subalgebra of the quantum affine $U_q(\hat{sl}(2)$ and implemented it to find a solution of the reflection equation. We have related this consideration to a model of nonequilibrium physics where the boundary operators generate a tridiagonal Askey–Wilson algebra, which is the linear covariance algebra of the bulk $U_q(\hat{s}u(2))$ symmetry. It is the hidden symmetry that allows for the exact solvability in the stationary state and provides the framework for employing Bethe ansatz to determine the dynamical properties of the open process.

Acknowledgments

The authors are grateful to V Rittenverg and R B Zhang for interest in the work and stimulating discussions. BA and PPK would like to thank the Department of Physical Sciences, University of Helsinki, and the Helsinki Institute of Physics for the hospitality during their visits in September–October 2007 when this work was completed. The support of the Academy of Finland and the Bulgarian Academy of Sciences under a joint research project is acknowledged.

References

[1] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435
[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 246
[3] Drinfeld V G *Proc. 1986 Congress of Math.* (Providence, RI: APS) p 798
[4] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[5] Chaichian M and Demichev A 1996 *Introduction to Quantum Groups* (Singapore: World Scientific)
[6] Kulish P P and Sklyanin E K 1982 *Lecture Notes Phys.* **151** 61
[7] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[8] Kulish P P and Sklyanin E K 1991 *J. Phys. A: Math. Gen.* **24** L435
[9] Mezincescu L and Nepomechie R I 1998 *Int. J. Mod. Phys.* A **13** 2747
[10] Delius G W and MacKay N J 2003 *Commun. Math. Phys.* **233** 173
[11] Doikou A 2005 *J. Stat. Mech.* P12005
[12] Nichols A, Rittenberg V and de Gier J 2005 *J. Stat. Mech.* P003
[13] Chari V and Pressley A 1991 *Commun. Math. Phys.* **142** 261
[14] Reshetikhin N Yu and Semenov–Tian–Shansky M A 1990 *Lett. Math. Phys.* **19** 133
[15] Ito T and Terwilliger P 2003 Tridiagonal pairs and the quantum affine algebra $U_q(\hat{s}l_2)$ *Preprint* 0310042
[16] Terwilliger P 2006 *Lecture Notes Math.* **1883** 225
[17] Terwilliger P 2003 *Preprint* math.QA/0307016
[18] Terwilliger P and Vidunas R 2004 *J. Algebra Appl.* **3** 411
[19] Dolan L and Grady M 1982 *Phys. Rev.* D **3** 1587
[20] Zhedanov A S 1991 *Theor. Math. Fiz.* **89** 190
[21] Koornwinder T 2007 *SIGMA* **3** 063
[22] Baseilhac P and Koizumi K 2005 *Nucl. Phys.* B **720** 325
[23] Granovskii Y A and Zhedanov A S 1993 *J. Phys. A: Math. Gen.* **26** L357
[24] Askey R A and Wilson J A 1985 *Mem. Am. Math. Soc.* **54** 1

[25] Jimbo M and Miwa T 1995 *Algebraic Analysis of Solvable Lattice Models* (Providence, RI: American Mathematical Society) p 152
[26] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[27] de Vega H J and Gonzalez-Ruiz A 1994 *J. Phys. A: Math. Gen.* **27** 6129
[28] Baseilhac P 2005 *Nucl. Phys.* B **709** 491
[29] Vidunas R 2005 *Preprint* math.RA/0505041
[30] Chaichian M and Kulish P P 2002 *Multiple Facets of Quantization and Supersymmetry* ed M Olshanetsky and A Vainstein (Singapore: World Scientific)
[31] Schuetz G M 2000 *Phase Transitions and Critical Phenomena* vol 19 (London: Academic)
[32] Schreckenberg M, Schadschneider A, Nagel K and Ito N 1995 *Phys. Rev.* E **51** 2939
[33] Macdonald J T, Gibbs J H and Pipkin A C 1968 *Biopolymers* **6** 1
[34] Krug J and Spohn H 1991 *Solids far from Equilibrium* ed C Godreche (Cambridge: Cambridge University Press)
[35] Krug J 1991 *Phys. Rev. Lett.* **67** 1882
[36] Derrida B 1998 *Phys. Rep.* **301** 65 and references therein
[37] Derrida B, Evans M R, Hakim V and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 1493
[38] Schuetz G M and Domany E 1993 *J. Stat. Phys.* **72** 277
[39] Sandow S 1994 *Phys. Rev.* E **50** 2660
[40] Essler F H L and Rittenberg V 1996 *J. Phys. A: Math. Gen.* **29** 3375
[41] Mallick K and Sandow S 1997 *J. Phys. A: Math. Gen.* **30** 4513
[42] Collnelli O and Mallick K 2006 *J. Phys. A: Math. Gen.* **39** 10647
Alcaraz F C and Lazo M J 2006 *J. Phys. A: Math. Gen.* **39** 11335
[43] Arndt P F, Heinzel T and Rittenberg V 1998 *J. Phys. A: Math. Gen.* **31** 833
[44] Isaev A, Pyatov P and Rittenberg V 2001 *J. Phys. A: Math. Gen.* **34** 5815
[45] Stinchcombe R B and Schuetz G M 1995 *Phys. Rev. Lett.* **75** 140
[46] Aneva B 2007 *J. Phys. A: Math. Theor.* **40** 11677
[47] Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524
[48] Chaichian M and Kulish P P 1990 *Phys. Lett.* B **234** 72
[49] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* A **23** L415
[50] Blythe R A, Evans M R, Colaiori F and Essler F H L 2000 *J. Phys. A: Math. Gen.* **33** 2313
[51] Sasamoto T 1999 *J. Phys. A: Math. Gen.* **32** 7109
[52] Uchliama M, Sasamoto T and Wadati M 2004 *J. Phys. A: Math. Gen.* **37** 4985
[53] Sandow S and Schuetz G 1994 *Europhys. Lett.* **26** 7
[54] Alcaraz F C, Dasmahapatra S and Rittenberg V 1998 *J. Phys. A: Math. Gen.* **31** 845
[55] de Gier J and Essler F 2005 *Phys. Rev. Lett.* **95** 240601
[56] de Gier J and Essler F H L 2006 *J. Stat. Mech.* P12011
[57] Arutyunov G, Frolov S and Staudacher M 2005 *J. High Energy Phys.* JHEP05(2005)054
[58] Beisert N and Staudacher M 2005 *Nucl. Phys.* B **727** 1